39. The finite element method for the Crank-Nicholson implementation ($M_x = (\delta, 1-2\delta, \delta)$) of the convection equation is given by

$$M_x \frac{f_j^{n+1} - f_j^n}{\Delta t} + \frac{1}{2} u \left(L_x f_j^{n+1} + L_x f_j^n \right) = 0$$

(a) Show that this yields the following equation for the temporal update

$$\left(\delta - \frac{1}{4}c\right) f_{j-1}^{n+1} + (1-2\delta) f_j^{n+1} + \left(\delta + \frac{1}{4}c\right) f_{j+1}^{n+1} = \left(\delta + \frac{1}{4}c\right) f_{j-1}^{n} + (1-2\delta) f_j^{n} + \left(\delta - \frac{1}{4}c\right) f_{j+1}^{n}$$

(b) Derive the amplification factor

$$g = \frac{(1-2\delta) + 2\delta \cos k\Delta_x - \frac{ic}{2} \sin k\Delta_x}{(1-2\delta) + 2\delta \cos k\Delta_x + \frac{ic}{2} \sin k\Delta_x}$$

(c) Demonstrate that this scheme is unconditionally stable for the convection equation.

40. Download the program trans.f for the transport equation from the web and go through the readme file. Note that you can gain additional insight into properties of any one of the following cases by using a smaller width of the truncated sine wave or by varying the diffusion coefficient.

(a) Run the program with the provided parameters (upwind method) for the truncated sine wave (with the default and half of the default width). Plot the results (with trans.pro). Estimate the diffusion (through the damping of the wave) and dispersion (through propagation speed) for your results. By identifying the parameters used for this run, are diffusion and dispersion consistent with the theoretical expectations and can you explain inconsistencies with numerical errors for diffusion and dispersion?

(b) Repeat the run with the default parameters for the Leapfrog method. Plot the results and compare these to the upwind scheme. What is the influence of increasing the diffusion coefficient to $\alpha = 0.1$?

(c) Repeat the default run for the Lax-Wendroff method. Vary the value of q to minimize the error. Plot the results for these cases. What is the corresponding value of q? Rerun the Leapfrog with $c = 1$. What do you observe?

(d) Repeat the default run for the finite difference Crank-Nicholson scheme ($\delta = 0$). By varying q, what is the value of q that minimizes the error? Plot the result for this value of q.

(e) Finally run the problem with the generalized finite element Crank-Nicholson scheme (δ adjustable). For which value of δ do you find the minimum error? Plot the result and compare the result with the other methods.

Comment your overall findings. What did you learn in this comparison?

Project Reminder: Complete your project and prepare for the project presentation. Turn in a project report with this homework.

This homework (problems 39 and 40) count for 70%. Please turn the final project report and the solutions to this homework on Monday, 5/6/2013