Conduct *3 hours; choose 3 of the 4 problems; Open book; Closed homework*

1. Boltzmann equation:

The Boltzmann equation for the distribution function $f(\mathbf{x},\mathbf{v},t)$ of a collisionless ionized gas is

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{\mathbf{F}}{m}\cdot\nabla_{\mathbf{v}} f = 0. \tag{1}$$

Multiply the equation with $\frac{m}{2}v^2$ and integrate the first two terms of the equation over velocity space to derive the corresponding terms for the energy equation. Mass density ρ, fluid velocity $\mathbf{u}$, and pressure tensor are defined as

$$\begin{aligned}
\rho(\mathbf{x},t) = mn(\mathbf{x},t) &= m\int_{\infty}^{\infty} d^3v f(\mathbf{x},\mathbf{v},t) \\
\mathbf{u}(\mathbf{x},t) &= \frac{1}{n}\int_{\infty}^{\infty} d^3v \mathbf{v} f(\mathbf{x},\mathbf{v},t) \\
\underline{\underline{\Pi}}(\mathbf{x},t) &= m\int_{\infty}^{\infty} d^3v\,(\mathbf{v}-\mathbf{u})\,(\mathbf{v}-\mathbf{u})\,f(\mathbf{x},\mathbf{v},t)
\end{aligned}$$

with $p(\mathbf{x},t) = \frac{1}{3}\mathrm{Tr}\underline{\underline{\Pi}}$, and Heatflux $\mathbf{L} = \frac{m}{2}\int d^3v\,(\mathbf{v}-\mathbf{u})^2\,(\mathbf{v}-\mathbf{u})\,f(\mathbf{r},\mathbf{v},t)$ (Make use of the substitution $\widetilde{\mathbf{v}} = (\mathbf{v}-\mathbf{u})$ for velocity integrals and note that $\int_{\infty}^{\infty} d^3v\,(\mathbf{v}-\mathbf{u})\,f(\mathbf{x},\mathbf{v},t) = 0$)

2. Characteristics:

Consider the general second order equation

$$\frac{\partial^2 u}{\partial t^2} + \lambda c\frac{\partial^2 u}{\partial t\partial x} + c^2\frac{\partial^2 u}{\partial x^2} = G$$

a) Transform this equation into an equivalent system of first order PDE's by introducing appropriate auxiliary variables $R = \partial u/\partial t$, etc.

b) Using this system of equations, show that the second order equation is hyperbolic for $\lambda^2 - 4 > 0$, parabolic for $\lambda^2 - 4 = 0$, and elliptic for $\lambda^2 - 4 < 0$.

c) What are the characteristics for the hyperbolic case?

3. Consistency:

The advection equation $\partial f/\partial t + v\partial f/\partial x = 0$ is discretized with the scheme

$$f_j^{n+1} - f_j^{n-1} = -\frac{4c}{3}\left(f_{j+1}^n - f_{j-1}^n\right) + \frac{c}{6}\left(f_{j+2}^n - f_{j-2}^n\right)$$

with $c = v\Delta t/\Delta x$.

What are the conditions for consistency? Expand the equation in a Taylor series to determine the leading error term. Is consistency satisfied?

4. Stability:

a) Using the von Neumann method determine the amplification factor for the discretization in Problem 3.

b) Discuss stability as a function of c and what it implies for Δt.