18. Force balance equation in cylindrical coordinates:

Assume \( \partial/\partial \theta = \partial/\partial z \) and \( B_r = 0 \) and show that the force balance equation in cylindrical coordinates \((r, \theta, z)\) takes the form

\[
\frac{\partial}{\partial r} \left( p + \frac{B_\theta^2 + B_z^2}{2\mu_0} \right) + \frac{B_\theta^2}{\mu_0 r} = 0
\]

**Solution:**

Force balance equation:

\[
-\nabla p + j \times B = 0
\]

in cylindrical coordinates with \( \partial/\partial \theta = \partial/\partial z = 0 \)

\[
\nabla p = \frac{\partial p}{\partial r} \hat{e}_r
\]

\[
\mu_0 \vec{j} = -\frac{\partial B_z}{\partial r} \hat{e}_\theta + \frac{1}{r} \frac{\partial r B_\theta}{\partial r} \hat{e}_z
\]

such that

\[
\vec{j} \times \vec{B} = \frac{1}{\mu_0} \left( \frac{\partial B_z}{\partial r} B_z - \frac{1}{r} \frac{\partial r B_\theta}{\partial r} B_\theta \right) \hat{e}_r
\]

\[
= \left[ -\frac{1}{2\mu_0} \frac{d}{dr} \left( B_z^2 + B_\theta^2 \right) - \frac{B_\theta^2}{\mu_0} \right] \hat{e}_r
\]

yielding the force balance equation in the radial direction:

\[
\frac{dp}{dr} + \frac{1}{2\mu_0} \frac{d}{dr} \left( B_z^2 + B_\theta^2 \right) + \frac{B_\theta^2}{\mu_0 r} = 0
\]

19. \( \theta \) pinch:

Assume a constant current \( j_0 \) in the \( z \) direction in a cylindrical coordinate system.

a) Compute the magnetic field \( B_\theta(r) \) and integrate the force balance equation to obtain \( p(r) \). The pressure at \( r = 0 \) is \( p_0 \).

b) Determine the critical radius for which the pressure decreases to 0.

c) Show that the equilibrium condition for the \( \theta \) pinch

\[
\frac{dp_0}{dr} = \frac{B_0}{\mu_0 r} \frac{d \left( r B_0 \right)}{dr}
\]
can be expressed as
\[
\frac{d \ln B_0}{d \ln r} = \beta \frac{d \ln p_0}{2 \ln r} - 1
\]

**Solution:**

(a) Magnetic field \(B_\theta(r)\):

\[
\frac{1}{r} \frac{\partial r B_\theta}{\partial r} = \mu_0 j_0
\]

\[
B_\theta = \frac{\mu_0 j_0}{r} \int_0^r r dz = \frac{\mu_0 j_0}{2} r
\]

Note \(B_\theta = 0\) at \(r = 0\) because \(r = 0\) is a singular point for \(B_\theta\).

Pressure:

\[
\frac{dp}{dr} = -j_z B_\theta
\]

which yields

\[
p(r) = -\frac{\mu_0 j_0^2}{2} \int r dz = -\frac{\mu_0 j_0^2}{4} r^2 + p_0
\]

such that \(p(0) = p_0\).

(b) The pressure assumes 0 for

\[
-\frac{\mu_0 j_0^2}{4} r_c^2 + p_0 = 0
\]

or

\[
r_c^2 = \frac{4 p_0}{\mu_0 j_0^2}
\]

(c) Show that

\[
\frac{dp_0}{dr} = B_0 \frac{d (r B_0)}{dr}
\]

is equivalent to

\[
\frac{d \ln B_0}{d \ln r} = \beta \frac{d \ln p_0}{2 \ln r} - 1
\]

with

\[
\frac{d}{d \ln r} = \left( \frac{d \ln r}{dr} \right)^{-1} \frac{d}{dr} = r \frac{d}{dr}
\]

\[
\frac{r}{B_0} \frac{dB_0}{dr} = \frac{2 \mu_0 p_0}{2 B_0^2} \frac{r}{p_0} \frac{dp_0}{dr} - 1 = \frac{\mu_0 r}{B_0^2} \frac{dp_0}{dr} - 1
\]

Multiplication with \(B_0^2/\mu_0\):

\[
\frac{dp_0}{dr} = \frac{B_0}{\mu_0} \frac{dB_0}{dr} + \frac{B_0^2}{\mu_0 r} \frac{dp_0}{dr} = \frac{B_0}{\mu_0} \frac{d (r B_0)}{dr}
\]