8. Pressure/energy equation:
Consider the pressure equation for isotropic pressure in the absence of heat conduction
\[
\frac{1}{\gamma - 1} \left( \frac{\partial p}{\partial t} + \nabla \cdot (p \mathbf{u}) \right) = -p \nabla \cdot \mathbf{u} + Q^E
\]

Assuming \( h = p/\rho^\gamma \), demonstrate that the pressure equation combined with the continuity equation can be written as
\[
\frac{\partial h}{\partial t} + \mathbf{u} \cdot \nabla h = R
\]
and compute \( R \) as a function of \( Q^E \).

**Solution:**
Assuming a function \( h = p\rho^{-\gamma} \), demonstrate that the pressure equation combined with the continuity equation can be written as
\[
\frac{\partial h}{\partial t} + \mathbf{u} \cdot \nabla h = R
\]

Using the continuity
\[
\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u}
\]
and the pressure equations
\[
\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{u} + (\gamma - 1) Q^E
\]
we obtain
\[
\frac{dh}{dt} = \rho^{-\gamma} \frac{dp}{dt} - \gamma \rho p^{-\gamma - 1} \frac{d\rho}{dt} = -\gamma \rho \rho^{-\gamma} \nabla \cdot \mathbf{u} + \gamma \rho \rho^{-\gamma} \nabla \cdot \mathbf{u} + (\gamma - 1) \rho^{-\gamma} Q^E = (\gamma - 1) Q^E / \rho^\gamma
\]

Note that for resistive heating \( Q^E = \eta j^2 \geq 0 \) such that \( dh/dt \geq 0 \). The quantity \( h \) is a measure of fluid entropy and increases only in the presence of nonadiabatic - for instance resistive heating.
9. a) Demonstrate that the two-dimensional compressible irrotational steady state flow
\[
\left(\frac{u^2}{a^2} - 1\right) \frac{\partial u}{\partial x} + \left(\frac{uv}{a^2}\right) \frac{\partial u}{\partial y} + \left(\frac{uv}{a^2}\right) \frac{\partial v}{\partial x} + \left(\frac{v^2}{a^2} - 1\right) \frac{\partial v}{\partial y} = 0
\]
\[-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0
\]
are equivalent to the equations
\[
-\frac{\gamma p}{\rho} \nabla \cdot \mathbf{u} + \mathbf{u} \cdot ((\mathbf{u} \cdot \nabla) \mathbf{u}) = 0
\]
(1)
\[
\nabla \times \mathbf{u} = 0
\]
(2)
for the velocity \(\mathbf{u} = (u, v)\) with \(a^2 = \gamma p/\rho\).

b) Derive equation (1) using the steady state continuity (2.18), momentum (2.19), and pressure equation (2.20) for isotropic pressure \(\Pi = p \frac{1}{\gamma}\) (no source terms, no external force terms, and no heat conduction). Hint: Use the continuity and momentum equations to replace the \nabla \rho and \nabla p terms in the pressure equation).

Solution:
(a) Eq. (1), 1st term:
\[
-\frac{\gamma p}{\rho} \nabla \cdot \mathbf{u} = -a^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)
\]
Eq. (1), 2nd term:
\[
\mathbf{u} \cdot ((\mathbf{u} \cdot \nabla) \mathbf{u}) = u \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}\right) u + v \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}\right) v
\]
\[
= u^2 \frac{\partial u}{\partial x} + uv \frac{\partial u}{\partial y} + uv \frac{\partial v}{\partial x} + v^2 \frac{\partial v}{\partial y}
\]
The sum of the two terms and division by \(a^2\) yields the first of the two equations.

2nd equation: The \(z\) component of the curl yields
\[
\mathbf{e}_z \cdot \nabla \times \mathbf{u} = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\]

(b) Derivation of equation (1) from:
\[
0 = -\nabla \cdot (\rho \mathbf{u})
\]
\[
0 = -\nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \nabla p
\]
\[
0 = -\nabla \cdot \left(\frac{1}{2} \rho u^2 \mathbf{u} + \frac{\gamma}{\gamma - 1} p \mathbf{u}\right)
\]
yield
\[
0 = -\rho (\mathbf{u} \cdot \nabla) \mathbf{u} - \mathbf{u} \nabla \cdot (\rho \mathbf{u}) - \nabla p
\]
\[
0 = -\frac{1}{2} \rho u^2 \nabla u^2 - \frac{1}{2} \mathbf{u} \nabla \cdot (\rho \mathbf{u}) - \frac{\gamma}{\gamma - 1} \rho \nabla \cdot \mathbf{u} - \frac{\gamma}{\gamma - 1} \mathbf{u} \cdot \nabla p
\]
With $\nabla \cdot (\rho \mathbf{u}) = 0$ we obtain

$$0 = -\rho (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p$$

$$0 = -\frac{1}{2} \rho \mathbf{u} \cdot \nabla u^2 - \frac{\gamma}{\gamma - 1} p \nabla \cdot \mathbf{u} - \frac{\gamma}{\gamma - 1} \mathbf{u} \cdot \nabla p$$

substituting $\nabla p$

$$-\frac{1}{2} \rho \mathbf{u} \cdot \nabla u^2 - \frac{\gamma}{\gamma - 1} p \nabla \cdot \mathbf{u} + \frac{\gamma}{\gamma - 1} \mathbf{u} \cdot (\rho (\mathbf{u} \cdot \nabla) \mathbf{u}) = 0$$

With $\nabla u^2 = 2 \mathbf{u} \times (\nabla \times \mathbf{u}) + 2 (\mathbf{u} \cdot \nabla) \mathbf{u}$ we obtain

$$-\frac{\gamma p}{\rho} \nabla \cdot \mathbf{u} + \mathbf{u} \cdot ((\mathbf{u} \cdot \nabla) \mathbf{u}) = 0$$
10. Read section 3.5 of the manuscript, derive the Fourier transform of the Navier Stokes equations, and show that the determinant for the Fourier transform matrix is

\[
(\sigma_x^2 + \sigma_y^2) \left[ i(u\sigma_x + v\sigma_y) + \frac{1}{Re} (\sigma_x^2 + \sigma_y^2) \right] = 0
\]

**Solution:**

Define the Fourier transform as

\[
\tilde{u}(\sigma_x, \sigma_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y) \exp(-i\sigma_x x) \exp(-i\sigma_y y) \, dx \, dy
\]

or symbolic as \( \tilde{u} = Fu \) with the property that

\[
i\sigma_x \tilde{u} = F \frac{\partial u}{\partial x} = \frac{\partial \tilde{u}}{\partial \sigma_x}, \quad i\sigma_y \tilde{u} = F \frac{\partial u}{\partial y} = \frac{\partial \tilde{u}}{\partial \sigma_y}
\]

Steady Navier-Stokes equations

\[
u_x + v_y = 0 \quad (3)
\]

\[
u u_x + v u_y + p_x - \frac{1}{Re} (u_{xx} + u_{yy}) = 0 \quad (4)
\]

\[
u v_x + v v_y + p_y - \frac{1}{Re} (v_{xx} + v_{yy}) = 0 \quad (5)
\]

Fourier transform:

\[
i\sigma_x \tilde{u} + i\sigma_y \tilde{v} = 0 \quad (6)
\]

\[
i\sigma_x \tilde{u} u + i\sigma_y \tilde{v} \tilde{u} + i\sigma_x \tilde{p} - \frac{1}{Re} (i\sigma_x)^2 \tilde{u} + (i\sigma_y)^2 \tilde{u} = 0 \quad (7)
\]

\[
i\sigma_x \tilde{u} \tilde{v} + i\sigma_y \tilde{v} \tilde{v} + i\sigma_y \tilde{p} - \frac{1}{Re} (i\sigma_x)^2 \tilde{v} + (i\sigma_y)^2 \tilde{v} = 0 \quad (8)
\]

In matrix form:

\[
\begin{bmatrix}
i\sigma_x & i\sigma_y & 0 \\
i(u\sigma_x + v\sigma_y) + \frac{1}{Re} (\sigma_x^2 + \sigma_y^2) & \sigma_x & \sigma_y \\
0 & i\sigma_x & i\sigma_y
\end{bmatrix}
\begin{bmatrix}
\tilde{u} \\
\tilde{v} \\
\tilde{p}
\end{bmatrix} = 0
\]

which yields from \( \det[.] = 0 \) the equation

\[
(\sigma_x^2 + \sigma_y^2) \left[ i(u\sigma_x + v\sigma_y) + \frac{1}{Re} (\sigma_x^2 + \sigma_y^2) \right] = 0
\]
11. General Technique - 2nd Derivative Approximation

a) Use the general technique to determine the coefficients $a$ to $c$ and the leading error term in the following expression

$$\frac{d^2 f}{dx^2} = af_{i-2} + bf_{i-1} + cf_i$$

b) Do the same for the expression

$$\frac{d^2 f}{dx^2} = af_{i-1} + bf_i + cf_{i+1} + df_{i+2} + ef_{i+3}$$

**Solution:**

(a) Taylor expansion of

$$\frac{d^2 f}{dx^2} \approx af_{i-2} + bf_{i-1} + cf_i = (a+b+c)f_i + (-2a-b)\Delta x \frac{df}{dx}\bigg|_i$$

$$+(4a+b)\frac{\Delta x^2}{2} \frac{d^2 f}{dx^2}\bigg|_i + (-8a-b)\frac{\Delta x^3}{6} \frac{d^3 f}{dx^3}\bigg|_i$$

$$+(16a-b)\frac{\Delta x^4}{24} \frac{d^4 f}{dx^4}\bigg|_i + . . .$$

Conditions for $a$, $b$, $c$, and $d$:

$$a + b + c = 0$$

$$-2a - b = 0$$

$$4a + b = 2/\Delta x^2$$

Solution

$$a = 1/\Delta x^2$$

$$b = -2/\Delta x^2$$

$$c = 1/\Delta x^2$$

such that

$$\frac{d^2 f}{dx^2} = \frac{f_{i-2} - 2f_{i-1} + f_i}{\Delta x^2} - \Delta x \frac{d^3 f}{dx^3}\bigg|_i + O(\Delta x^2)$$

(b) Taylor expansion of

$$\frac{d^2 f}{dx^2} \approx af_{i-1} + bf_i + cf_{i+1} + df_{i+2} + ef_{i+3}$$

$$= (a + b + c + d + e)f_i + (-a + c + 2d + 3e)\Delta x \frac{df}{dx}\bigg|_i$$

$$+(a + c + 4d + 9e)\frac{\Delta x^2}{2} \frac{d^2 f}{dx^2}\bigg|_i + (-a + c + 8d + 27e)\frac{\Delta x^3}{6} \frac{d^3 f}{dx^3}\bigg|_i$$

$$+(a + c + 16d + 81e)\frac{\Delta x^4}{24} \frac{d^4 f}{dx^4}\bigg|_i + (-a + c + 32d + 243e)\frac{\Delta x^5}{120} \frac{d^5 f}{dx^5}\bigg|_i$$
Conditions for $a$, $b$, $c$, and $d$:

\[
\begin{align*}
    a + b + c + d + e &= 0 \\
    -a + c + 2d + 3e &= 0 \\
    a + c + 4d + 9e &= 2/\Delta x^2 \\
    -a + c + 8d + 27e &= 0 \\
    a + c + 16d + 81e &= 0
\end{align*}
\]

eliminating $a$ from (2) to (5) (by adding 2 and 3, 3 and 4, 4 and 5):

\[
\begin{align*}
    2c + 6d + 12e &= 2/\Delta x^2 \\
    2c + 12d + 36e &= 2/\Delta x^2 \\
    2c + 24d + 108e &= 0
\end{align*}
\]

eliminating $c$ (by subtracting 1 from 2 and from 3) in the above equations:

\[
\begin{align*}
    6d + 24e &= 0 \\
    9d + 48e &= -/\Delta x^2
\end{align*}
\]

Solution:

\[
\begin{align*}
    e &= -1/12\Delta x^2 \\
    d &= 4/12\Delta x^2 \\
    c &= 6/12\Delta x^2 \\
    a &= 11/12\Delta x^2 \\
    b &= -20/12\Delta x^2
\end{align*}
\]

and

\[
-a + c + 32d + 243e = -120/12\Delta x^2
\]

such that

\[
\frac{d^2 f}{dx^2} = \frac{11f_{i-1} - 20f_i + 6f_{i+1} + 4f_{i+2} - f_i}{12\Delta x^2} - \frac{\Delta x^3}{12} \frac{d^5 f}{dx^5} + O(\Delta x^4)
\]

The order of the error for the 3 point approximation in part (a) is linear in $\Delta x$ and for the 5 point approximation in part (b) it is $\Delta x^3$. In general for the $n$th derivative at least $n+1$ grid points are needed for the approximation. For an $m$ point approximation the error is usually of order $\Delta x^{m-n}$. 