26. Stability of the linear FEM for the diffusion equation

Using linear finite elements on a uniform grid result in the following expression for the one-dimensional diffusion equation

\[ M_s \Delta T_j^{n+1} - \alpha \Delta t L_{xx} \left[ (1 - \beta) T_j^n + \beta T_j^{n+1} \right] = 0 \]

with the mass operator \( M_s = (1/6, 2/3, 1/6) \), the second derivative operator \( L_{xx} = (1, -2, 1) / \Delta x^2 \), and \( \Delta T_j^{n+1} = T_j^{n+1} - T_j^n \). Consider \( 0 \leq \beta \leq 1 \).

(a) Derive the discretized equation and show that the amplification factor for the von Neumann stability analysis is

\[ g = \frac{\left( \frac{2}{3} - 2s (1 - \beta) \right) + 2 \left( \frac{1}{6} + s (1 - \beta) \right) \cos (k \Delta x)}{\left( \frac{2}{3} + 2s \beta \right) + 2 \left( \frac{1}{6} - s \beta \right) \cos (k \Delta x)} \]

(b) Determine the stability properties for the parameter \( s \).

(Hint: It can be helpful to distinguish the cases \( \beta > 1/2 \) and \( \beta < 1/2 \))

Solution:

(a) Discretized equation:

\[
M_s \Delta T_j^{n+1} = \alpha \Delta t L_{xx} \left[ (1 - \beta) T_j^n + \beta T_j^{n+1} \right]
\]

\[
\frac{1}{6} \Delta T_{j-1}^{n+1} + \frac{2}{3} \Delta T_j^{n+1} + \frac{1}{6} \Delta T_{j+1}^{n+1} = \frac{1}{6} \left( T_{j-1}^n - 2T_j^n + T_{j+1}^n \right) + s \left( 1 - \beta \right) \left( T_{j-1}^n - 2T_j^n + T_{j+1}^n \right)
\]

\[
\left( \frac{2}{3} + 2s \beta \right) T_j^{n+1} + \left( \frac{1}{6} - s \beta \right) \left( T_{j-1}^{n+1} + T_{j+1}^{n+1} \right) = \left( \frac{2}{3} - 2s (1 - \beta) \right) T_j^n + \left( \frac{1}{6} + s (1 - \beta) \right) \left( T_{j-1}^n + T_{j+1}^n \right)
\]

Substituting \( T_j^n = \tilde{T} \exp [i(\omega t_n + k x_j)] \) with the relations \( g = T_j^{n+1} / T_j^n \) for the amplification factor, \( \exp (ik \Delta x) = T_{j+1}^n / T_j^n \) and \( \left( T_{j-1}^n + T_{j+1}^n \right) / T_j^n = (\exp (-ik \Delta x) + \exp (ik \Delta x)) = 2 \cos (k \Delta x) \) we divide the equation by \( T_j^n \):

\[
g \left[ \left( \frac{2}{3} + 2s \beta \right) + 2 \left( \frac{1}{6} - s \beta \right) \cos (k \Delta x) \right] = \left( \frac{2}{3} - 2s (1 - \beta) \right) + 2 \left( \frac{1}{6} + s (1 - \beta) \right) \cos (k \Delta x)
\]

or

\[
g = \frac{\left( \frac{2}{3} - 2s (1 - \beta) \right) + 2 \left( \frac{1}{6} + s (1 - \beta) \right) \cos (k \Delta x)}{\left( \frac{2}{3} + 2s \beta \right) + 2 \left( \frac{1}{6} - s \beta \right) \cos (k \Delta x)}
\]
(b) Stability $-1 \leq g \leq 1$:

(1) Case $-1 \leq g$:

$$-1 \leq \left(\frac{\frac{2}{3} - 2s(1-\beta)}{\frac{2}{3} + 2s\beta} + 2\left(\frac{1}{6} + s(1-\beta)\right)\cos(k\Delta x)\right)$$

$$-\left(\frac{1}{3} + s\beta\right) - \left(\frac{1}{6} - s\beta\right)\cos(k\Delta x) \leq \left(\frac{1}{3} - s + s\beta\right) + \left(\frac{1}{6} + s - s\beta\right)\cos(k\Delta x)$$

$$-\left(\frac{2}{3} + 2s\beta - s\right) \leq \left(\frac{1}{3} + s - 2s\beta\right)\cos(k\Delta x)$$

$$-\frac{2}{3} + s(1-2\beta) \leq \left(\frac{1}{3} + s(1-2\beta)\right)\cos(k\Delta x)$$

i) Consider first $\frac{1}{3} + s(1-2\beta) \geq 0$ or $s(1-2\beta) \geq -1/3$ which is always satisfied for $\beta \leq 1/2$ (For $\beta > 1/2$ this implies $s(2\beta - 1) \leq 1/3$).

$$-\frac{2}{3} + s(1-2\beta) \leq \left(\frac{1}{3} + s(1-2\beta)\right)\cos(k\Delta x) \leq -\frac{1}{3} - s(1-2\beta)$$

$$s(1-2\beta) \leq 1/6$$

With $s \leq \frac{1}{6(1-2\beta)}$ for $\beta < 1/2$

and no stability limit for $\beta \geq 1/2$ (but the condition to evaluate the inequality requires $s(2\beta - 1) \leq 1/3$).

ii) Now consider $\frac{1}{3} + s(1-2\beta) < 0$ or $s(1-2\beta) < -1/3$ which requires $\beta \geq 1/2$ and implies $s(2\beta - 1) > 1/3$

$$-\frac{2}{3} + s(1-2\beta) \leq \left(\frac{1}{3} + s(1-2\beta)\right)\cos(k\Delta x) \leq \frac{1}{3} + s(1-2\beta)$$

This is always satisfied and required the range $s(2\beta - 1) > 1/3$.

Combining the results from (i) and (ii) the inequality $-1 \leq g$ is satisfied for all $s$ if $\beta \geq 1/2$ and requires $s \leq 1/6(1-2\beta)$ for $\beta < 1/2$.

(2) Case $g \leq 1$

$$\left(\frac{1}{3} - s(1-\beta)\right) + \left(\frac{1}{6} + s(1-\beta)\right)\cos(k\Delta x) \leq \left(\frac{1}{3} + s\beta\right) + \left(\frac{1}{6} - s\beta\right)\cos(k\Delta x)$$

$$-s + s\cos(k\Delta x) \leq 0$$

$$-s(1 - \cos(k\Delta x)) \leq 0$$

This is satisfied for all $s$ (note $s$ must be positive) such that no additional stability limits result.

In summary the scheme is unconditionally stable for $\beta \geq 1/2$ and requires $s \leq 1/6(1-2\beta)$ for $\beta < 1/2$. 

2
27. Obtain solutions using the program duct on an $11 \times 11$ grid for decreasing values of $b/a$ until the centerline solution across the smaller dimension is within 1% rms of the one-dimensional parabolic profile $u = u_0(1 - y^2)$ with $u_0 = 0.5$.

**Solution:**

Problem 24 demonstrates that the centerline flow for $b/a = 1$ is about 0.29, i.e., different from 0.5 as expected for the 1D solution. Following are the results for $b/a = 0.374$ and 0.15. The dashed lines give the parabolic profile.

The 1% rms error of the one-dimensional parabolic is reached for the left plot. It demonstrates a considerable distortion from the solution for $b/a = 1$. The right plots show a solution for even a smaller value of $b/a = 0.15$. In this solution it is rather obvious that the solution for the most part appears one-dimensional except close to the boundaries in $x$. (Note: the plots are actually for a $21^2$ grid in which case the 1% error is obtained for $b/a = 0.367$)

**Background:** Smaller values of $b/a$ imply that the $x$ derivative term becomes less important, i.e., the solution should become more one-dimensional. The original differential equation was the same in $x$ and $y$ only the aspect ratio of the computational domain (width in $x$ and width $y$) is changed. The smaller dimension ($y$) starts to dominate the solution because the boundaries in $y$ are much closer to the interior domain than the boundaries in $x$. 

![Plot](image-url)
28. Viscous flow in a rectangular duct is governed by $(b/a)^2 \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + 1 = 0$ subject to the boundary conditions $w = 0$ at $x = \pm 1, y = \pm 1$. The exact solution for this problem is given by

$$w = \left(\frac{8}{\pi^2}\right)^2 \sum_{i=1,3,5,..}^{L} \sum_{j=1,3,5,..}^{L} \left[ \frac{(-1)^{(i+j)/2-1}}{ij ((ib/a)^2 + j^2)} \cos(0.5i\pi x) \cos(0.5j\pi y) \right]$$

with sufficiently large $L$. As an approximate solution, choose

$$w = \sum_{j=1}^{N} a_j(1 - x^2)^j(1 - y^2)^j.$$

Obtain approximate solutions using the subdomain method with $N = 1$ and 2 (use the domains $|x|, |y| \leq 1$ and $|x|, |y| \leq a$ with $a = \sqrt{1/2}$). Compare these with the exact solution $L = 21$. Comment your results.

Solution:

Trial functions:

$$w = \sum_{j=1}^{N} a_j(1 - x^2)^j(1 - y^2)^j$$

For $N = 1$: $w = a_1(1 - x^2)(1 - y^2)$

$$\frac{\partial^2 w}{\partial x^2} = -2a_1(1 - y^2) \quad \frac{\partial^2 w}{\partial y^2} = -2a_1(1 - x^2)$$

Residual: $R_1 = -2a_1 \left[ r_a(1 - y^2) + (1 - x^2) \right] + 1$

For $N = 2$: $w = a_1(1 - x^2)(1 - y^2) + a_2(1 - x^2)^2(1 - y^2)^2$

$$\frac{\partial^2 w}{\partial x^2} = -2a_1(1 - y^2) - 4a_2(1 - 3x^2)(1 - y^2)^2$$
$$\frac{\partial^2 w}{\partial y^2} = -2a_1(1 - x^2) - 4a_2(1 - x^2)^2(1 - 3y^2)$$

Residual:

$$R_2 = -2a_1 \left[ r_a(1 - y^2) + (1 - x^2) \right] - 4a_2 \left[ r_a(1 - 3x^2)(1 - y^2)^2 + (1 - x^2)^2(1 - 3y^2) \right] + 1$$

Subdomain Method Method:

Weight functions for the residual integral are step functions which are equal to 1 in the chosen subdomain and 0 otherwise. For $N=1$ we choose the entire domain:

i) Integral for $N = 1$:

$$\int_{-a}^{a} \int_{-a}^{a} \left\{ -2a_1 \left[ r_a(1 - y^2) + (1 - x^2) \right] + 1 \right\} \, dxdy = 0$$

4
or
\[2a_1\int_{-a}^{a} \int_{-a}^{a} [r_a(1 - y^2) + (1 - x^2)] \, dxdy = \int_{-a}^{a} \int_{-a}^{a} dxdy\]
\[2a_1 2a_2 \left(1 - \frac{a^2}{3}\right)(r_a + 1) = 4a^2\]

With the integrals below this yields: \(2a_1 (r_1 I_1 + I_1) = 4\)
\[> a_1 = \frac{1}{2} \left(1 - \frac{a^2}{3}\right)(r_a + 1)^{-1}\] (1)
\[\text{or } a_1 = \frac{3}{4} (r_a + 1)^{-1} \quad \text{for } a = 1\] (2)
\[\text{or } a_1 = \frac{3}{5} (r_a + 1)^{-1} \quad \text{for } a = \sqrt{\frac{1}{2}}\] (3)

**ii) Integrals for \(N = 2\):** The choice of the subdomains is not fixed such that the result is not unique. Here we choose a square with \(|x|, |y| \leq a\) keeping \(a\) variable for the moment (later choosing \(a = 1\) and \(1/\sqrt{2}\)):

**Domain \(a\):**
\[2a_1 \int_{-a}^{a} \int_{-a}^{a} [r_a(1 - y^2) + (1 - x^2)] \, dxdy\]
\[+ 4a_2 \int_{-a}^{a} \int_{-a}^{a} [r_a(1 - 3x^2)(1 - y^2)^2 + (1 - x^2)^2(1 - 3y^2)] \, dxdy = \int_{-a}^{a} \int_{-a}^{a} dxdy\]
or
\[2a_1 \left[r_a 2a_2 \left(1 - \frac{a^2}{3}\right) + 2a_2 \left(1 - \frac{a^2}{3}\right)\right] +\]
\[+ 4a_2 \left[r_a 2a(1 - a^2)2a \left(1 - \frac{2}{3}a^2 + \frac{1}{5}a^4\right) + 2a(1 - a^2)2a \left(1 - \frac{2}{3}a^2 + \frac{1}{5}a^4\right)\right] = 4a^2\]
\[=> \quad \left(1 - \frac{a^2}{3}\right) a_1 + 2 (1 - a^2) \left(1 - \frac{2}{3}a^2 + \frac{1}{5}a^4\right) a_2 = \frac{1}{2} (r_a + 1)^{-1}\](4)

We obtain two equations to determine \(a_1\) and \(a_2\) from (4) for two different choices of \(a\). Using \(a = 1\) yields again
\[a_1 = \frac{3}{4} (r_a + 1)^{-1}\]
and substitution in \(a_2\):
\[a_2 = \frac{1}{2(1 - a^2)} \left(1 - \frac{2}{3}a^2 + \frac{1}{5}a^4\right)^{-1} \left[\frac{1}{2} (r_a + 1)^{-1} - \left(1 - \frac{a^2}{3}\right) a_1\right] = \frac{1}{2(1 - a^2)} \left(1 - \frac{2}{3}a^2 + \frac{1}{5}a^4\right)^{-1} \left(1 - a^2\right) (r_a + 1)^{-1}\]
\[= -\frac{1}{8} \left(1 - \frac{2}{3}a^2 + \frac{1}{5}a^4\right)^{-1} (r_a + 1)^{-1}\]
With \( a = 1/\sqrt{2} \) the equations for \( a_1 \) and \( a_2 \) are

\[
\begin{align*}
    a_1 &= \frac{3}{4} (r_a + 1)^{-1} \\
    a_2 &= -\frac{15}{86} (r_a + 1)^{-1}
\end{align*}
\]  

(5)  

(6)

Graphics for the various methods:

Analytic solution:

Subdomain method:
Integrals used in the computation of the residual integrals:

\[
I_1 = \int_{-a}^{a} (1 - x^2) \, dx = 2a \left[ 1 - \frac{a^2}{3} \right]
\]

\[
I_2 = \int_{-a}^{a} (1 - x^2)^2 \, dx = \int_{-a}^{a} (1 - 2x^2 + x^4) \, dx = 2a \left[ 1 - \frac{2}{3}a^2 + \frac{a^4}{5} \right]
\]

\[
I_3 = \int_{-a}^{a} (1 - 3x^2) \, dx = 2a \left[ 1 - a^2 \right]
\]