Problem 1. Second order moments of the first two terms in the Boltzmann equation:

\[ \frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{F}{m} \cdot \nabla v f = 0 \]

Definitions with \( \rho = mn(\mathbf{r},t) \):

\[ n(\mathbf{r},t) = \int d^3v f(\mathbf{r},v,t) \]  
\[ u(\mathbf{r},t) = \frac{1}{n(\mathbf{r},t)} \int d^3v f(\mathbf{r},v,t) \]  
\[ \Pi(\mathbf{r},t) = m \int d^3v (v - u)(v - u)f(\mathbf{r},v,t) \]

All integrals are from \(-\infty\) to \(+\infty\) over the three velocity components.

i) Term (1): With \( \tilde{v} = (v - u) \) or \( v = \tilde{v} + u \)

\[ I_1 = \frac{m}{2} \int d^3v \tilde{v}^2 \frac{\partial f}{\partial t} = \frac{m}{2} \int d^3\tilde{v} \tilde{v} \cdot (\tilde{v} + u) f \]
\[ = \frac{m}{2} \int d^3v \tilde{v}^2 + 2\tilde{v} \cdot u + u^2 f \]
\[ = \frac{m}{2} \int d^3v \tilde{v}^2 f + m \frac{\partial}{\partial t} \left( u \cdot \int d^3v f \right) + m \frac{\partial}{\partial t} \left( u^2 \int d^3v f \right) \]

The first term with the definition of \( \tilde{v} \) yields the diagonal terms of the pressure tensor

\[ \frac{m}{2} \frac{\partial}{\partial t} \int d^3v \tilde{v}^2 f = \frac{1}{2} \frac{\partial}{\partial t} \left( \Pi_{xx} + \Pi_{yy} + \Pi_{zz} \right) = \frac{1}{2} \frac{\partial}{\partial t} \text{Tr}\Pi = \frac{3}{2} \frac{\partial \rho}{\partial t} \]

The second term is 0 because of the definition of \( u \). The last term yields

\[ \frac{m}{2} \frac{\partial}{\partial t} \left( u^2 \int d^3v f \right) = \frac{m}{2} \frac{\partial}{\partial t} \left( nu^2 \right) = \frac{1}{2} \frac{\partial \rho u^2}{\partial t} \]

such that

\[ I_1 = \frac{3}{2} \frac{\partial \rho}{\partial t} + \frac{1}{2} \frac{\partial \rho u^2}{\partial t} \]

ii) Term (2):

\[ I_2 = \frac{m}{2} \int d^3v \tilde{v} \cdot \nabla f(\mathbf{r},v,t) = \frac{m}{2} \nabla \cdot \int d^3v \tilde{v} f(\mathbf{r},v,t) \]
\[ = \frac{m}{2} \nabla \cdot \int d^3v (\tilde{v} + u)^2 (\tilde{v} + u) f \]
\[ I_2 = \frac{m}{2} \sum_{i,j} \left( \frac{\partial}{\partial x_i} \int d^3\tilde{v} \left( \tilde{v}_i^2 + 2\tilde{v}_i\tilde{v}_j u_j + \tilde{v}_i^2 - 2u_i\tilde{v}_j u_j - u_i u_j^2 \right) f \right) \]

With the heat flux \( L = \frac{m}{2} \int d^3v (v - u)^2 (v - u) f(v, v, t) \) the first term becomes

\[ I_2 = \frac{m}{2} \sum_{i,j} \left( \frac{\partial}{\partial x_i} \int d^3\tilde{v} \left( \tilde{v}_i^2 + 2\tilde{v}_i\tilde{v}_j u_j + \tilde{v}_i^2 - 2u_i\tilde{v}_j u_j - u_i u_j^2 \right) f \right) \]

In vector representation using \( \sum_j \Pi_{jj} = \text{Tr} \Pi \) this becomes

\[ I_2 = \nabla \cdot L + \nabla \cdot (\Pi \cdot u) + \frac{3}{2} \nabla \cdot (pu) + \frac{1}{2} \nabla \cdot (\rho u^2) \]

iii) Combining the two terms:

\[ I_1 + I_2 = \frac{3}{2} \frac{\partial p}{\partial t} + \frac{1}{2} \frac{\partial \rho u^2}{\partial t} + \nabla \cdot L + \nabla \cdot (\Pi \cdot u) + \frac{3}{2} \nabla \cdot (pu) + \frac{1}{2} \nabla \cdot (\rho u^2) \]
Problem 2.

(a) General second order equation
\[ \frac{\partial^2 u}{\partial t^2} + \lambda c \frac{\partial^2 u}{\partial t \partial x} + c^2 \frac{\partial^2 u}{\partial x^2} = G \]

Introducing \( R = \partial u/\partial t \) and \( S = \partial u/\partial x \) one obtains
\[ \begin{align*}
\frac{\partial R}{\partial t} + \lambda c \frac{\partial R}{\partial x} + c^2 \frac{\partial S}{\partial x} &= -G \\
\frac{\partial S}{\partial t} - \frac{\partial S}{\partial x} &= 0
\end{align*} \]

(b) Nontrivial solutions require
\[ \det \begin{pmatrix} 0 & \lambda_t + \lambda c \lambda_x & c^2 \lambda_x \\ \lambda_t & 0 & 0 \\ 0 & \lambda_x & -\lambda_t \end{pmatrix} = 0 \]

\[ \Rightarrow \]
\[ c^2 \lambda_x^2 \lambda_t + \lambda_t^2 (\lambda_t + \lambda c \lambda_x) = 0 \]

Division by \( \lambda_t^3 \) and defining \( r = \lambda_x/\lambda_t \) (one can equivalently define \( r = \lambda_t/\lambda_x \) to get \( dx/dt \) in part c) yields
\[ c^2 r^2 + \lambda c r + 1 = 0 \]

or
\[ r^2 + \frac{\lambda}{c} r + \frac{\lambda^2}{4c^2} = \frac{\lambda^2 - 4}{4c^2} \]

with the solutions
\[ r = \frac{-\lambda}{2c} \pm \frac{1}{2c} \sqrt{\lambda^2 - 4} \]

There are two real solutions for \( \lambda^2 - 4 > 0 \) implying a hyperbolic PDE, one solution for \( \lambda^2 - 4 = 0 \Rightarrow \) parabolic PDE, and no real solution for \( \lambda^2 - 4 < 0 \) implying an elliptic PDE.

(c) The vector \((\lambda_x, \lambda_t)\) represents the normal to a curve given by \((x,t)\) and is therefore equivalent to \((dt, -dx)\) or \(r = -dt/dx\) such that ** back to equation for \(r^2\)
\[ c^2 - \lambda c \frac{dx}{dt} + \left( \frac{dx}{dt} \right)^2 = 0 \]

\[
\left( \frac{dx}{dt} - \frac{\lambda c}{2} \right)^2 = \frac{\lambda^2 c^2}{4} - c^2
\]

\[
\frac{dx}{dt} = \frac{\lambda c}{2} \pm \sqrt{\frac{\lambda^2 c^2}{4} - c^2} = \frac{\lambda c}{2} \left[ 1 \pm \sqrt{1 - \frac{4}{\lambda^2}} \right]
\]

for the characteristics.
Problem 3.

Consistency requires that the discretized algebraic equation converges to the actual partial differential equation \( \partial f / \partial t + v \partial f / \partial x = 0 \) in the limit of \( \Delta t, \Delta x \to 0 \).

Inserting the definition \( c = v \Delta t / \Delta x \) the discretized equation is

\[
\frac{f_{j}^{n+1} - f_{j}^{n-1}}{2\Delta t} = \frac{-4v}{3} \frac{1}{2\Delta x} \left( f_{j+1}^{n} - f_{j-1}^{n} \right) + \frac{v}{3} \frac{1}{4\Delta x} \left( f_{j+2}^{n} - f_{j-2}^{n} \right)
\]

Taylor expansion of the left side of the equation

\[
\frac{f_{j}^{n+1} - f_{j}^{n-1}}{2\Delta t} = \frac{1}{2\Delta t} \left\{ (1 - 1) f_{j}^{n} + (1 + 1) \Delta t f_{t,j}^{n} + \frac{1}{2} (1 - 1) \Delta t^2 f_{tt,j}^{n} + \frac{1}{6} (1 + 1) \Delta t^3 f_{ttt,j}^{n} \right. \\
\left. \quad + \frac{1}{24} (1 - 1) \Delta t^4 f_{tttt,j}^{n} + \frac{1}{120} (1 + 1) \Delta t^5 f_{ttttt,j}^{n} \right\}
\]

\[
= f_{t,j}^{n} + \frac{1}{6} \Delta t^2 f_{tt,j}^{n} + \frac{1}{120} \Delta t^4 f_{tttt,j}^{n}
\]

The Taylor expansion for the first term on the right is the same replacing \( t \) with \( x \)

\[
\frac{1}{2\Delta x} \left( f_{j+1}^{n} - f_{j-1}^{n} \right) = f_{x,j}^{n} + \frac{1}{6} \Delta x^2 f_{xxx,j}^{n} + \frac{1}{120} \Delta x^4 f_{xxxx,j}^{n}
\]

For the second term on the right we need to replace \( \Delta x \) with \( 2\Delta x \)

\[
\frac{1}{4\Delta x} \left( f_{j+2}^{n} - f_{j-2}^{n} \right) = f_{x,j}^{n} + \frac{4}{6} \Delta x^2 f_{xxx,j}^{n} + \frac{16}{120} \Delta x^4 f_{xxxx,j}^{n}
\]

Multiplication of these equations with \(-4v/3\) and with \( v/3 \) and adding the results yields

\[
RHS = -vf_{x,j}^{n} - \frac{v}{30} \Delta x^4 f_{xxxx,j}^{n}
\]

Finally using \( f_t = -vf_x \) and \( f_{tt} = -vf_{xt} = v^2 f_{xx} \) and \( f_{ttt} = v^2 f_{xxt} = -v^3 f_{xxx} \) and the definition \( v\Delta t = c\Delta x \)

\[
\frac{f_{j}^{n+1} - f_{j}^{n-1}}{2\Delta t} = f_{t,j}^{n} - \frac{1}{6} \Delta t^2 v^3 f_{xxx,j}^{n} - \frac{1}{120} \Delta t^4 v^5 f_{xxxx,j}^{n} + ..
\]

\[
= f_{t,j}^{n} - \frac{v}{6} c^2 \Delta x^2 f_{xxx,j}^{n} - \frac{v}{120} c^4 \Delta x^4 f_{xxxx,j}^{n} + ..
\]

And combining the different Taylor expansions

\[
f_{t,j}^{n} + vf_{x,j}^{n} = \frac{v}{6} c^2 \Delta x^2 f_{xxx,j}^{n} + \frac{v}{120} c^4 \Delta x^4 f_{xxxx,j}^{n} - \frac{v}{30} \Delta x^4 f_{xxxx,j}^{n} + O \left( \Delta x^6 \right)
\]

\[
= \frac{v}{6} \Delta x^2 \left[ c^2 f_{xxx,j}^{n} + \left( \frac{c^4 - 4}{20} \right) \Delta x^2 f_{xxxx,j}^{n} \right]
\]

Such that despite the higher accuracy of the spatial derivative the leading error term \( \frac{v}{6} \Delta x^2 c^2 f_{xxx,j}^{n} \) is determined by the temporal derivative. Therefore the scheme satisfies consistency but is accurate only up to 2nd order in \( \Delta x \).
Problem 4. Is the scheme

\[ f_{j}^{n+1} - f_{j}^{n-1} = -\frac{4c}{3} (f_{j+1}^{n} - f_{j-1}^{n}) + \frac{c}{6} (f_{j+2}^{n} - f_{j-2}^{n}) \]

with \( c = v \Delta t / \Delta x \) stable?

(a) Amplification factor: Substituting the amplification factor \( g = f_{j}^{n+1} / f_{j}^{n} \) and \( f_{j}^{n+1} = f_{j}^{n} \exp[ik\Delta x] \) yields

\[
\frac{g - 1}{g} = -\frac{4c}{3} \left[ \exp(ik\Delta x) - \exp(-ik\Delta x) \right] + \frac{c}{6} \left[ \exp(i2k\Delta x) - \exp(-i2k\Delta x) \right]
\]

\[ = -\frac{8c}{3}ic \sin(k\Delta x) + \frac{c}{3}ic \sin(2k\Delta x) \]

\[ = \frac{c}{3}i[8 \sin(k\Delta x) + 2 \sin(2k\Delta_x) \cos(k\Delta_x)] = 2ih(k\Delta_x) \]

with \( h(k\Delta_x) = \frac{c}{3} \sin(k\Delta_x) [4 + \cos(k\Delta_x)] \) which yields for \( g \):

\[
g^2 - 2g ih - 1 = 0 \quad \text{or} \quad (g - ih)^2 = 1 - h^2
\]

\[ \implies g = ih \pm \sqrt{1 - h^2} \]

(b) Stability: Since \( g \) is complex we consider the absolute value of \( g \)

(1) Assuming \( h^2 \leq 1 \), i.e., \(-1 \leq h \leq 1\) the magnitude of the amplification factor is given by

\[ |g|^2 = gg^* = h^2 + (1 - h^2) = 1, \]

i.e., the scheme is stable in the range \(-1 \leq h \leq 1\).

(2) Assuming \( h^2 > 1 \), i.e., \( h > 1 \) or \( h < -1 \):

\[
g^2 = i^2 \left( h \pm \sqrt{h^2 - 1} \right)^2 = -\left[ 2h^2 \pm 2h \sqrt{h^2 - 1} - 1 \right]
\]

Choosing the ' + ' sign and \( h > 1 \):

\[
g^2 = -\left[ 2h^2 + 2h \sqrt{h^2 - 1} - 1 \right] < - \left[ h^2 + 2h \sqrt{h^2 - 1} \right] < -1.
\]

For \( h < -1 \) and the ' - ' sign:

\[
g^2 = -\left[ 2h^2 - 2h \sqrt{h^2 - 1} - 1 \right] < - \left[ h^2 - 2h \sqrt{h^2 - 1} \right] < -1,
\]

i.e., the scheme is unstable for \( h > 1 \) and \( h < -1 \). Therefore overall stability requires \(-1 \leq h \leq 1\). To identify the stable range of \( c \) is given by

\[ \left| \frac{c}{3} \sin(k\Delta_x) [4 + \cos(k\Delta_x)] \right| \leq 1 \]
or with $x = k\Delta x$ and $f = \sin x (4 + \cos x)$ the stable range is $|c| \leq 3 / \max f$. The maxima and minima of $f(x)$ are given by $df/dx = \cos x (4 + \cos x) - \sin x (\sin x) = 2\cos^2 x + 4\cos x - 1 = 0$ or

$$\cos x = -1 + \sqrt{3/2} \quad \sin x = \pm \sqrt{-\frac{3}{2} + \sqrt{6}}$$

$$\Rightarrow \quad \max (f(x)) = \sqrt{\frac{3}{2} \left(3 + \sqrt{3/2}\right)} \approx \pm 4.12 \quad \text{and}$$

$$|c| \leq \min \left(\frac{3}{|\sin x(4 + \cos x)|}\right) = \frac{3}{\max f} \approx 0.73$$

A cruder estimate using $\sin x \leq 1$ and $4 + \cos x \leq 5$ yields $|c| \leq 3/5 = 0.6$. 